give us your opinion on the usefulness or not of this kind of meetings for the development of an area, in particular in an area that constitutes a small intersection point of several others.

Valentina.—Numerical semigroups is apparently a narrow subject, but it gathers people from different areas.

Thus, the most interesting talks in the meeting were for me the talks where also other subjects, e.g. from commutative algebra or from code theory, appear.

It was also interesting and useful to meet personally some mathematicians who worked on similar problems than me and that I know only though their papers.

The meeting was also pleasant because there are not people that consider themselves big stars, as sometime happens, and there was a very nice cooperative atmosphere.

José Carlos.—These meetings are very useful since they encourage the grouping of mathematicians interested in the study and applications of numerical semigroups coming from all over the world. This provides a contact at first hand with the latest advances in this field. It also provides discussions between different researchers that could not happen otherwise.

Ralf.—It is very important to have a chance to meet people in this relatively narrow area. This gives possibilities to talk about problems, but still to get views from different angles.

Scott.—I think it is highly useful. Most of the participants at the IMNS meetings found numerical semigroups by working in some other area. In my case, it was Commutative Algebra. In other cases, it was Computer Science, Graph Theory, Algebraic Geometry, ... The list is almost endless. I am not so sure that the intersection mentioned above is so small. In fact, I think it has grown drastically over the past 10 years and I believe that attendance at the next IMNS meetings will exceed that of any of the first three editions of this congress.

The multivariate extremal index and tail dependence

by Helena Ferreira*

Abstract.—If we obtain a tail dependence coefficient of the common distribution of the vectors in a multivariate stationary sequence then we do not have necessarily the correspondent coefficient of the limiting multivariate extreme value model. In opposition to sequences of independent and identically distributed random vectors, the local clustering of extremes allowed by stationarity can increase or decrease the tail dependence.

The temporal dependence at extreme levels can be summarized by the function multivariate extremal index and its effect in the tail dependence is well illustrated with Multivariate Maxima of Moving Maxima processes.

Keywords.—multivariate moving maxima, multivariate extremal index, tail dependence, multivariate extreme value distribution.

1. Introduction

For a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ with continuous marginal distributions $F_1, \ldots, F_d$ and copula C, let the bivariate (upper) tail dependence coefficients $\lambda_{jj'}^{(\mathbf{X})} \equiv \lambda_{jj'}^{(C)}$ be defined by

$$\lim_{u \uparrow 1} P\left(F_j(X_j) > u | F_{j'}(X_{j'}) > u\right), \quad (1)$$

for $1 \leq j < j' \leq d$. Tail dependence coefficients measure the probability of extreme values occurring for one random variable given that another assumes an extreme value too. Positive values correspond to tail dependence and null values mean tail independence. These coefficients can be defined via copulas and it holds that

$$\lambda_{jj'}^{(C)} = 2 - \lim_{u \uparrow 1} \frac{\ln C_{jj'}(u,u)}{\ln u}, \quad (2)$$

where $C_{jj'}$ is the copula of the sub-vector $(X_j, X_{j'})$ ([4], [7]).

Let F be a multivariate distribution function with continuous marginals, which is in the domain of attraction of a Multivariate Extreme Value (henceforth MEV) distribution $\hat{H}$ with standard Fréchet margins, that is, $F^n(nx_1, \ldots, nx_d) \underset{n\to\infty}{\longrightarrow} \hat{H}(x_1, \ldots, x_d)$with marginal distributions $\hat{H}(x_j) = \exp\left(-x_j^{-1}\right)$, $x_j > 0$, $j = 1, \ldots, d$. It is known that any bivariate tail dependence coefficient of F is the same as the corresponding coefficient of $\hat{H}$ ([8]).

Let $\{\mathbf{Y}_n\}_{n\geq 1}$ be a multivariate stationary sequence such that $F_{\mathbf{Y}_n} = F$ and $\mathbf{M}_n = (M_{n1}, \ldots, M_{nd})$ is the vector of componentwise maxima from $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$. If $\lim_{n\to\infty} P(M_{n1} \leq nx_1, \ldots, M_{nd} \leq nx_d) = H(x_1, \ldots, x_d)$, for some MEV distribution H, one question that naturally arises is Does dependence across the sequence affect the bivariate tail dependence of the limiting MEV? In other words, we would like to know the relation between the tail dependence coefficients of the limiting MEV H and the limiting MEV

$$\hat{H}(x_1, \ldots, x_d) = \lim_{n\to\infty} P\left(\hat{M}_{n1} \leq nx_1, \ldots, \hat{M}_{nd} \leq nx_d\right) = \lim_{n\to\infty} F^n(nx_1, \ldots, nx_d),$$

where $\hat{\mathbf{M}}_n = (\hat{M}_{n1}, \ldots, \hat{M}_{nd})$ is the vector of pointwise maxima for a sequence of i.i.d. random vectors $\{\hat{\mathbf{Y}}_n\}_n \geq 1$ associated to $\{\mathbf{Y}_n\}_n \geq 1$, that is, such that, $F_{\hat{\mathbf{Y}}_n} = F_{\mathbf{Y}_n} = F$.

Our main purpose is to compare the bivariate tail dependence coefficients for the margins of the two Multivariate Extreme Value distributions H and $\hat{H}$ through the function multivariate extremal index ([6]), which resumes temporal dependence in $\{\mathbf{Y}_n\}_{n\geq 1}$.

We recall that the d–dimensional stationary sequence $\{\mathbf{Y}_n\}_{n\geq 1}$ is said to have a multivariate extremal index $\theta(\boldsymbol{\tau}) \in [0,1]$, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d) \in \mathbb{R}_+^d$, if for each $\boldsymbol{\tau}$ in $\mathbb{R}_+^d$,

* Department of Mathematics — University of Beira Interior, Covilhã

there exists levels $\mathbf{u}_n^{(\tau)} = (u_{n1}^{(\tau_1)}, \dots, u_{nd}^{(\tau_d)})$, $n \geq 1$, satisfying

$$nP(Y_{1j} > u_{nj}^{(\tau_j)}) \underset{n\to\infty}{\longrightarrow} \tau_j,\ j = 1, \dots, d,$$

$$P(\hat{\mathbf{M}}_n \leq u_n^{(\tau)}) \underset{n\to\infty}{\longrightarrow} \hat{\gamma}(\boldsymbol{\tau}) > 0$$

and

$$P(\mathbf{M}_n \leq \mathbf{u}_n^{(\tau)}) \underset{n\to\infty}{\longrightarrow} \hat{\gamma}(\boldsymbol{\tau})^{\theta(\boldsymbol{\tau})}.$$

The multivariate extremal index, although dependent of $\boldsymbol{\tau}$, is an homogeneous function of order zero and if it exists for $\{\mathbf{Y}_n\}_{n\geq 1}$ then any sequence of sub-vectors $\{(\mathbf{Y}_n)_A\}_{n\geq 1}$ with indices in $A \subset \{1, \dots, d\}$ has multivariate extremal index

$$\theta_A(\boldsymbol{\tau}_A) = \lim_{\substack{\tau_i \to 0^+ \\ i \notin A}} \theta(\tau_1, \dots, \tau_d),\ \ \boldsymbol{\tau}_A \in \mathbb{R}_+^{|A|}.$$

Moreover, for each marginal sequence $\{Y_{n,j}\}_{n\geq 1}$, the extremal index $\theta_{\{j\}} \equiv \theta_j$ is constant.

Under suitable long range dependence conditions for the stationary sequence $\{\mathbf{Y}_n\}_{n\geq 1}$, like strong-mixing condition, and for $r_n = o(n)$ we have

$$\frac{1}{\theta(\boldsymbol{\tau})} = \lim_{n\to+\infty} E\left(\sum_{k=1}^{r_n} \mathbb{I}_{\{\mathbf{Y}_k \not\leq \mathbf{u}_n^{(\tau)}\}} \middle| \sum_{k=1}^{r_n} \mathbb{I}_{\{\mathbf{Y}_k \not\leq \mathbf{u}_n^{(\tau)}\}} > 0\right),$$

where $\mathbb{I}_A$ denotes the indicator function. That is, the extremal index is the reciprocal of the limiting mean of the cluster size of exceedances of $\mathbf{u}_n^{(\tau)}$.

As motivation, we first compare in the next section the bivariate tail dependence coefficients of $\hat{H}$ and H arising in two Multivariate Maxima Moving Maxima (henceforth M4) processes ([12]), where we can compute directly these coefficients. Next we present the main result with a corollary on the M4 processes.

Since bivariate tail dependence for MEV distributions is related with the extremal coefficients ([13], [11]), we translate the main result in terms of these coefficients.

Finally, in the last section we discuss the multivariate generalizations of the results.

2. M4 examples

Let $\{Z_{l,n}\}_{l\geq 1, -\infty<n<\infty}$ be an array of independent random vectors with standard Fréchet margins. A multivariate maxima of moving maxima process is defined by

$$Y_{n,j} = \bigvee_{l\geq 1} \bigvee_{-\infty<k<\infty} \alpha_{lkj} Z_{l,n-k},\ j = 1, \dots, d, \qquad (3)$$

$n \geq 1$, where $\{\alpha_{lkj}, l \geq 1, -\infty < k < \infty, 1 \leq j \leq d\}$, are nonnegative constants satisfying

$$\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \alpha_{lkj} = 1 \quad \text{for } j = 1, \dots, d.$$

Extremal behaviour of these processes was developed in [12], where it is proved that the sequence $\{\mathbf{Y}_n\}_{n\geq 1}$ has multivariate extremal index

$$\theta(\tau_1, \dots, \tau_d) = \frac{\displaystyle\sum_{l=1}^{\infty} \bigvee_{-\infty\leq k\leq+\infty} \bigvee_{j=1,\dots,d} \alpha_{lkj}\tau_j}{\displaystyle\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} \bigvee_{j=1,\dots,d} \alpha_{lkj}\tau_j}.$$

and the extremal index of $\{Y_{nj}\}_{n\geq 1}$ is

$$\theta_j = \sum_{l=1}^{\infty} \bigvee_{-\infty<k<\infty} \alpha_{lkj},\ \ j = 1, \dots, d.$$

The M4 class of processes, which are very flexible models for temporally dependent processes, plays a remarkable role in the multivariate extreme value theory since the multivariate extremal index of a stationary max-stable sequence $\{\mathbf{Y}_n\}_{n\geq 1}$ may be approximated uniformly by the multivariate extremal index of an M4 sequence ([1]).

The common copula $C_{\mathbf{Y}}$ of $\mathbf{Y}_n = (Y_{n,1}, \dots, Y_{n,d})$ is defined by

$$C_{\mathbf{Y}}(u_1, \dots, u_d) = \prod_{l=1}^{\infty}\prod_{k=-\infty}^{\infty} \left(u_1^{\alpha_{lk1}} \wedge \dots \wedge u_d^{\alpha_{lkd}}\right),$$

$u_j \in [0, 1]$, $j = 1, \dots, d$. The dependence across the d dimensions is regulated by the structure of the signatures a_{lkj} in each l–th moving pattern. We will consider two examples with a finite number l of moving patterns and finite range $k_1 \leq k \leq k_2$ for its signatures.

Example 1.—We first consider a M4 process with one moving pattern and finite range $-1 \leq k \leq 1$ for the sequence dependence, defined as follows. For $n \geq 1$, let

$$\begin{cases} Y_{n,1} = \frac{1}{8}Z_{1,n-1} \vee \frac{1}{8}Z_{1,n} \vee \frac{6}{8}Z_{1,n+1} \\ Y_{n,2} = \frac{2}{8}Z_{1,n-1} \vee \frac{1}{8}Z_{1,n} \vee \frac{5}{8}Z_{1,n+1} \end{cases}$$

We have in this case,

$$C_{\mathbf{Y}}(u_1, u_2) \equiv C_{\mathbf{Y}_n}(u_1, u_2) =$$
$$\left(u_1^{1/8} \wedge u_2^{2/8}\right)\left(u_1^{1/8} \wedge u_2^{1/8}\right)\left(u_1^{6/8} \wedge u_2^{5/8}\right)$$

and

$$\lambda^{(C_{\mathbf{Y}})} = \lambda^{(\hat{C})} = 2 - \left(\frac{2}{8} + \frac{1}{8} + \frac{6}{8}\right) = \frac{7}{8},$$

where $\hat{C}$ denotes the copula of $\hat{H}$. Otherwise

$$H(x_1, x_2) = \exp\left(-\left(\frac{6x_1^{-1}}{8} \vee \frac{5x_2^{-1}}{8}\right)\right).$$

Therefore $C(u_1, u_2) = u_1 \wedge u_2$ and $\lambda^{(C)} = 1 > \lambda^{(\hat{C})}$.

Example 2.—We will now consider a modification in the above example through the introduction of one more pattern. Let, for each $n \geq 1$,

$$\begin{cases} Y_{n,1} = \frac{1}{8}Z_{1,n} \vee \frac{6}{8}Z_{1,n+1} \vee \frac{1}{8}Z_{2,n} \\ Y_{n,2} = \frac{1}{8}Z_{1,n} \vee \frac{5}{8}Z_{1,n+1} \vee \frac{2}{8}Z_{2,n} \end{cases}.$$

We have the same $C_{\mathbf{Y}}$ and $\lambda^{(\hat{C})} = 7/8$ as in the previous example, but here

$$H(x_1, x_2) =$$
$$\exp\left(-\left(\frac{6x_1^{-1}}{8} \vee \frac{5x_2^{-1}}{8}\right)\right)\exp\left(-\left(\frac{x_1^{-1}}{8} \vee \frac{2x_2^{-1}}{8}\right)\right)$$

and therefore

$$C(u_1, u_2) = \left(u_1^{6/7} \wedge u_2^{5/7}\right)\left(u_1^{1/7} \wedge u_2^{2/7}\right).$$

Then $\lambda^{(C)} = 2 - (6/7 + 2/7) = 6/7 < \lambda^{(\hat{C})}$.

These examples show that the dependence structure of the sequence can increase or decrease the tail dependence coefficients in the limiting MEV model. In the next proposition we will quantify such variation through the function multivariate extremal index.

3. Main result

In the this section we will relate $\lambda_{jj'}^{(C)}$ with $\lambda_{jj'}^{(\hat{C})}$, where C is the copula of the MEV distribution H and $\hat{C}$ corresponds to the limiting MEV distribution for the associated i.i.d. sequence. We first remark that it follows from the definition of the multivariate extremal index

$$\hat{H}\left(\frac{1}{\tau_j}, \frac{1}{\tau_{j'}}\right) = \hat{C}_{jj'}\left(e^{-\tau_j}, e^{-\tau_{j'}}\right),$$

and

$$H\left(\frac{1}{\tau_j}, \frac{1}{\tau_{j'}}\right) = C_{jj'}\left(e^{-\tau_j\theta_j}, e^{-\tau_{j'}\theta_{j'}}\right).$$

Then

$$C_{jj'}\left(e^{-\tau_j\theta_j}, e^{-\tau_{j'}\theta_{j'}}\right) = \left(\hat{C}_{jj'}\left(e^{-\tau_j}, e^{-\tau_{j'}}\right)\right)^{\theta(\tau_j, \tau_{j'})},$$

where $\theta(\tau_j, \tau_{j'})$ is the bivariate extremal index of $\{(Y_{nj}, Y_{nj'})\}_{n\geq 1}$. Therefore, for each $(u_j, u_{j'}) \in [0, 1]^2$, we have

$$C_{jj'}(u_j, u_{j'}) = \left(\hat{C}_{jj'}\left(u_j^{1/\theta_j}, u_{j'}^{1/\theta_{j'}}\right)\right)^{\theta\left(-\frac{\ln u_j}{\theta_j}, -\frac{\ln u_{j'}}{\theta_{j'}}\right)}. \qquad (4)$$

This relation enables us to compare the tail dependence coefficients $\lambda_{jj'}^{(C)}$ with $\lambda_{jj'}^{(\hat{C})}$ through the function $\theta(\tau_j, \tau_{j'})$.

Proposition 3.1.—If C and $\hat{C}$ satisfy (4) then, for each $1 \leq j < j' \leq d$, it holds that

(a) $\lambda_{jj'}^{(C)} = 2 + \theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right) \ln \hat{C}_{jj'}\left(e^{-1/\theta_j}, e^{-1/\theta_{j'}}\right)$,

(b) $$\lambda_{jj'}^{(C)} = \lambda_{jj'}^{(\hat{C})} + \ln \frac{\hat{C}_{jj'}\left(e^{-\theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right)/\theta_j}, e^{-\theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right)/\theta_{j'}}\right)}{\hat{C}_{jj'}\left(e^{-1}, e^{-1}\right)}.$$

Proof.—From the spectral measure representation of $\hat{H}$ ([2]), the copula $\hat{C}$ can be written as follows:

$$\hat{C}(u_1, \dots, u_d) = \exp\left(-\int_{\mathcal{S}_d} \bigvee_{j=1}^{d} w_j(-\ln u_j)\, d\hat{W}(\mathbf{w})\right),$$

where $\hat{W}$ is a finite measure on the unit shere $\mathcal{S}_d$ of $\mathbb{R}_+^d$ satisfying $\int_{\mathcal{S}_d} w_j\, d\hat{W}(\mathbf{w}) = 1$, $j = 1, \dots, d$. Then

$$\lambda_{jj'}^{(\hat{C})} = 2 - \lim_{u\uparrow 1} \frac{\ln \hat{C}_{jj'}(u, u)}{\ln u} =$$
$$2 - \int_{\mathcal{S}_d} (w_j \vee w_{j'})\, d\hat{W}(\mathbf{w}) =$$
$$2 + \ln \hat{C}_{jj'}(e^{-1}, e^{-1}). \qquad (5)$$

By using (4) and the homogeneity of order zero of the multivariate extremal index, it follows that

$$\lambda_{jj'}^{(C)} = 2 - \lim_{u\uparrow 1} \frac{\ln C_{jj'}(u, u)}{\ln u} =$$
$$-\theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right) \lim_{u\uparrow 1} \frac{\int_{\mathcal{S}_d} \left(\frac{-\ln u w_j}{\theta_j} \vee \frac{-\ln u w_{j'}}{\theta_{j'}}\right) d\hat{W}(\mathbf{w})}{-\ln u} =$$
$$-\theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right) \int_{\mathcal{S}_d} \left(\frac{w_j}{\theta_j} \vee \frac{w_{j'}}{\theta_{j'}}\right) d\hat{W}(\mathbf{w}) =$$
$$2 + \theta\left(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}\right) \ln \hat{C}_{jj'}\left(e^{-1/\theta_j}, e^{-1/\theta_{j'}}\right). \qquad (6)$$

To obtain the second statement we combine (a) and the max-stability of $\hat{C}_{jj'}$. ⊣

The previous result is valid to any MEV copulas C and $\hat{C}$ copulas related by a multivariate extremal index $\theta(\tau_1, \dots, \tau_d)$, that is, satisfying the relation

$$C(u_1, \dots, u_d) =$$
$$\left(\hat{C}\left(u_1^{1/\theta_1}, \dots, u_d^{1/\theta_d}\right)\right)^{\theta\left(-\frac{\ln u_1}{\theta_1}, \dots, -\frac{\ln u_d}{\theta_d}\right)}, \qquad (7)$$

$(u_1, \dots, u_d) \in [0, 1]^d$, which leads to (4).

For the particular case of M4 processes it is known ([3]) that

$$\lambda_{jj'}^{(\hat{C})} = 2 - \sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} (\alpha_{lkj} \vee \alpha_{lkj'}),$$

which is in general greater than zero. We add now, as a corollary of the previous proposition, the expression of $\lambda_{jj'}^{(C)}$ for these processes.

Corollary 3.2.—Let $\{\mathbf{Y}_n\}_{n\geq 1}$ be a M4 process defined as in (2). Then, for any $1 \leq j < j' \leq d$, it holds that

$$\text{(a)}\ \lambda_{jj'}^{(C)} = 2 - \sum_{l=1}^{\infty}\left(\frac{\bigvee\limits_{-\infty\leq k\leq+\infty} \alpha_{lkj}}{\theta_j} \vee \frac{\bigvee\limits_{-\infty\leq k\leq+\infty} \alpha_{lkj'}}{\theta_{j'}} \right),$$

(b) if $l = 1$ then $\lambda_{jj'}^{(C)} = 1$,

(c) $\lambda_{jj'}^{(C)} > \lambda_{jj'}^{(\hat{C})}$ if and only if

$$\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\left(\alpha_{lkj} \vee \alpha_{lkj'}\right) >$$

$$\sum_{l=1}^{\infty}\left(\bigvee_{-\infty\leq k\leq+\infty} \alpha_{lkj}/\theta_j \vee \bigvee_{-\infty\leq k\leq+\infty} \alpha_{lkj'}/\theta_{j'} \right).$$

We can apply the previous formula in (a) and find the same results in the previous examples.

As we have seen, it is easy to construct examples for which $\lambda_{jj'}^{(C)} > \lambda_{jj'}^{(\hat{C})}$ or $\lambda_{jj'}^{(C)} < \lambda_{jj'}^{(\hat{C})}$. However, if $\lambda_{jj'}^{(\hat{C})} = 1$ then $\lambda_{jj'}^{(C)} = 1$, as a consequence of the Proposition 3.1 and the Corollary 2 in [1], which states that $\lambda_{jj'}^{(\hat{C})} = 1$ leads to $\theta(\tau_j, \tau_{j'}) = \theta_j \wedge \theta_{j'}$. In fact, as a consequence of this result, we will have in (a) of the Proposition 3.1.,

$$\lambda_{jj'}^{(C)} = 2 - \frac{\theta(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}})}{\theta_j \wedge \theta_{j'}} = 2 - \frac{\theta_j \wedge \theta_{j'}}{\theta_j \wedge \theta_{j'}} = 2 - 1 = 1.$$

Another popular dependence coefficient, which is closely related to the multivariate extemal index, is the extremal coefficient. It is a summary coefficient for the extremal dependence introduced by Tiago de Oliveira ([13]) for bivariate extreme value distributions and extended to MEV distributions in [11]. Let the extremal coefficient of the MEV copula C be the constant ϵ_C such that $C(u, \ldots, u) = u^{\epsilon_C}$, for all $u \in [0, 1]$.

Since

$$-\log C(u_1, \ldots, u_d) =$$

$$\theta\left(-\frac{\ln u_1}{\theta_1}, \ldots, -\frac{\ln u_d}{\theta_d}\right) \int_{\mathscr{S}_d} \bigvee_{j=1}^{d} \frac{-\ln u_j w_j}{\theta_j}\, d\hat{W}(\mathbf{w})\}, \quad (8)$$

we get the following relation between the extremal coefficient of C, the copula $\hat{C}$ and the multivariate extremal index.

Proposition 3.3.—If C and $\hat{C}$ satisfy (7) then

$$\epsilon_C = -\theta\left(-\frac{1}{\theta_1}, \ldots, -\frac{1}{\theta_d}\right) \ln \hat{C}\left(e^{-1/\theta_1}, \ldots, e^{-1/\theta_d}\right).$$

This result extends in a natural way the known relation

$$\epsilon_{\hat{C}} = -\ln \hat{C}\,(e^{-1}, \ldots, e^{-1})$$

and enables to see the Proposition 3.1-(a) as an extension of the classical result $\lambda = 2 - \epsilon$ ([10]).

4. Multivariate tail dependence coefficients

How to characterise the strength of extremal dependence with respect to a particular subset of random variables of $\mathbf{X}$? One can use conditional orthant tail probabilities of $\mathbf{X}$ given that the components with indices in the subset J are extreme. The tail dependence of bivariate copulas can be extended as done in [9] and [5].

For $\varnothing \neq J \subset D = \{1, \ldots, d\}$, let $\lambda_J^{(\mathbf{X})} \equiv \lambda_J^{(C)} =$

$$\lim_{u\uparrow 1} P\left(\bigcap_{j\notin J} F_j(X_j) > u \,|\, \bigcap_{j\in J} F_j(X_j) > u\right). \quad (9)$$

If for some $\varnothing \neq J \subset \{1, \ldots, d\}$ the coefficient $\lambda_J^{(C)}$ exists and is positive then we say that $\mathbf{X}$ is (upper) orthant tail dependent. The relation (2) between the tail dependence coefficient and the bivariate copula can also be generalized by

$$\lambda_J^{(C)} = \lim_{u\uparrow 1} \frac{\sum\limits_{\varnothing\neq S\subset D} (-1)^{|S|-1} \ln C_S(\mathbf{u}_S)}{\sum\limits_{\varnothing\neq S\subset J} (-1)^{|S|-1} \ln C_S(\mathbf{u}_S)}, \quad (10)$$

where C_S denotes de copula of $\mathbf{X}_S$ and $\mathbf{u}_S$ the $|S|$-dimensional vector $(u, \ldots, u)$.

If $F^n(nx_1, \ldots, nx_d) \to \hat{H}(x_1, \ldots, x_d)$ then any positive tail dependence coefficient $\lambda_J^{(F)}$ of F is the same as the corresponding tail dependence coefficient $\lambda_J^{(\hat{C})}$ of the limiting MEV $\hat{H}$ ([5]). In the case of $\theta(\boldsymbol{\tau}) \neq 1$, the limiting MEV model does not preserve orthant tail dependence coefficients.

Let $\theta_S^* \equiv \theta_S((1/\theta_1, \ldots, 1/\theta_d)_S)$ denotes the multivariate extremal index of the sequence of sub-vectors $\{(\mathbf{Y}_n)_S\}_{n\geq 1}$ in the sub-vector of $\theta(1/\theta_1, \ldots, 1/\theta_d)$ with components in S.

By using (8) and (10), we can compute $\lambda_J^{(C)}$ from the copula $\hat{C}$ as follows.

Proposition 4.1.—If C and $\hat{C}$ satisfy (7) then

$$\lambda_J^{(C)} = \frac{\sum\limits_{\varnothing\neq S\subset D} (-1)^{|S|-1} \theta_S^* \ln \hat{C}_S\left(\left(e^{-1/\theta_1}, \ldots, e^{-1/\theta_d}\right)_S\right)}{\sum\limits_{\varnothing\neq S\subset J} (-1)^{|S|-1} \theta_S^* \ln \hat{C}_S\left(\left(e^{-1/\theta_1}, \ldots, e^{-1/\theta_d}\right)_S\right)},$$

provided the ratio is defined.

We will illustrate the above proposition with the M4 processes.

Example 3.—Let $\{\mathbf{Y}_n\}_n \geq 1$ defined by

$$\begin{cases} Y_{n,1} = \frac{1}{8}Z_{1,n-1} \vee \frac{3}{8}Z_{1,n} \vee \frac{4}{8}Z_{1,n+1} \\ Y_{n,2} = \frac{2}{8}Z_{1,n-1} \vee \frac{2}{8}Z_{1,n} \vee \frac{4}{8}Z_{1,n+1} \\ Y_{n,3} = \frac{4}{8}Z_{1,n-1} \vee \frac{2}{8}Z_{1,n} \vee \frac{2}{8}Z_{1,n+1} \end{cases}$$

We have

$$C_{\mathbf{Y}}(u_1, u_2, u_3) = \left(u_1^{1/8} \wedge u_2^{2/8} \wedge u_3^{4/8}\right)$$
$$\left(u_1^{3/8} \wedge u_2^{2/8} \wedge u_3^{2/8}\right)\left(u_1^{4/8} \wedge u_2^{4/8} \wedge u_3^{2/8}\right),$$

$$\theta\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_3}\right) = \frac{4}{11}, \ \theta_{\{1,2\}}\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}\right) = \frac{4}{9},$$

$$\theta_{\{1,3\}}\left(\frac{1}{\theta_1}, \frac{1}{\theta_3}\right) = \frac{4}{11} \text{ and } \theta_{\{2,3\}}\left(\frac{1}{\theta_2}, \frac{1}{\theta_3}\right) = \frac{4}{10}.$$

Therefore

$$\lambda_{\{1,2\}}^{(C_Y)} = \lambda_{\{1,2\}}^{(\hat{C})} = \frac{-3 + \frac{9}{8} + \frac{11}{8} + \frac{10}{8} - \frac{10}{8}}{-2 + \frac{9}{8}} = \frac{4}{7}$$

and

$$\lambda_{\{1,2\}}^{(C)} = \frac{-3 + 1 + 1 + 1 - \frac{10}{11}}{-2 + 1} = \frac{10}{11} > \lambda_{\{1,2\}}^{(\hat{C})}.$$

Exemplo 4.—Let us consider now

$$\begin{cases} Y_{n,1} = \frac{1}{8}Z_{1,n} \vee \frac{6}{8}Z_{1,n+1} \vee \frac{1}{8}Z_{2,n} \\ Y_{n,2} = \frac{1}{8}Z_{1,n} \vee \frac{5}{8}Z_{1,n+1} \vee \frac{2}{8}Z_{2,n} \\ Y_{n,3} = \frac{1}{8}Z_{1,n} \vee \frac{7}{8}Z_{1,n+1} \end{cases}$$

We have

$$C_Y(u_1, u_2, u_3) = \left(u_1^{1/8} \wedge u_2^{1/8} \wedge u_3^{1/8}\right)$$
$$\left(u_1^{6/8} \wedge u_2^{5/8} \wedge u_3^{7/8}\right)\left(u_1^{1/8} \wedge u_2^{2/8}\right),$$

$$\theta\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_3}\right) = \frac{9}{10}, \ \theta_{\{1,2\}}\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}\right) = \frac{8}{9},$$

$$\theta_{\{1,3\}}\left(\frac{1}{\theta_1}, \frac{1}{\theta_3}\right) = 1 \text{ and } \theta_{\{2,3\}}\left(\frac{1}{\theta_2}, \frac{1}{\theta_3}\right) = \frac{9}{10}.$$

Therefore

$$\lambda_{\{2,3\}}^{(C_{\mathbf{Y}})} = \lambda_{\{2,3\}}^{(\hat{C})} = \frac{-3 + \frac{9}{8} + \frac{9}{8} + \frac{10}{8} - \frac{10}{8}}{-2 + \frac{10}{8}} = 1$$

and

$$\lambda_{\{2,3\}}^{(C)} = \frac{-3 + \frac{8}{7} + \frac{9}{7} + \frac{9}{7} - \frac{9}{7}}{-2 + \frac{9}{7}} = \frac{4}{5} < 1 = \lambda_{\{2,3\}}^{(\hat{C})}.$$

Examples show that dependence across the sequence can increase or decrease the tail dependence coefficients and $\lambda_J^{(\hat{C})} = 1$ does not imply that $\lambda_J^{(C)} = 1$ with J consisting of more than one index. Finally, Proposition 3.1 can be extended as

$$\lambda_{\{i\}}^{(C)} = \sum_{\varnothing\neq S\subset D} (-1)^{|S|-1} \theta_S^* \ln \hat{C}_S\left(\left(e^{-1/\theta_1}, \ldots, e^{-1/\theta_d}\right)_S\right) =$$

$$\lambda_{\{i\}}^{(\hat{C})} + \sum_{\varnothing\neq S\subset D} (-1)^{|S|-1} \ln \frac{\hat{C}_S\left(\left(e^{-\theta_S^*/\theta_1}, \ldots, e^{-\theta_S^*/\theta_d}\right)_S\right)}{\hat{C}_S\left(\left(e^{-1}, \ldots, e^{-1}\right)_S\right)},$$

and a theoretical comparison between $\lambda_J^{(C)}$ and $\lambda_J^{(\hat{C})}$ follows from the relation

$$\lambda_J^{(C)} = \frac{\lambda_{\{i\}}^{(C)}}{\lambda_{\{i\}}^{(C_J)}}, \lambda_{\{i\}}^{(C_J)} \neq 0.$$

References

[1] Ehlert, A. and Schlather, M. (2008). Capturing the multivariate extremal index: bounds and interconnections, Extremes, 11, 353–377.

[2] Haan, L. de and Resnick, S.I. (1977). Limit theory for multivariate sample extremes. Z. Wahrsch. Verw. Gebiete 40, 317–337.

[3] Heffernan, J. E., Tawn, J. A., Zhang, Z. (2007). Asymptotically (in)dependent multivariate maxima of moving maxima processes, Extremes, 10, 57–82.

[4] Joe, H. (1997). Multivariate Models and Dependence Concepts. Chapman & Hall, London.

[5] Li, H. (2009). Orthant tail dependence of multivariate extreme value distributions, J. Multivariate Anal., 100(1), 243–256.

[6] Nandagopalan, S. (1990). Multivariate extremes and estimation of the extremal index. Ph.D. Dissertation, Department of Statistics, University of North Carolina, USA.

[7] Nelsen, R.B. (2006). An Introduction to Copulas. Springer, New York.

[8] Resnick, S.I. (1987). Extreme Values, Regular Variation and Point Processes. Springer, New York.

[9] Schmid, F., Schmidt, R. (2007). Multivariate conditional versions of Spearman's rho and related measures of tail dependence. J. Multivariate Anal., 98, 1123–1140.

[10] Sibuya, M. (1960). Bivariate extreme statistics. Ann. Inst. Statist. Math. 11, 195-210.

[11] Smith, R.L. (1990). Max-stable processes and spatial extremes. Preprint, Univ. North Carolina, USA.

[12] Smith, R.L., Weissman, I. (1996). Characterization and estimation of the multivariate extremal index. Technical Report, Univ. North Carolina, USA.

[13] Tiago de Oliveira, J. (1962/63). Structure theory of bivariate extremes, extensions. Est. Mat., Estat. e Econ. 7, 165– 195.