Numerical semigroups problem list

by M. Delgado*, P. A. García-Sánchez** and J. C. Rosales***

1. Notable elements and first problems

A numerical semigroup is a subset of \( \mathbb{N} \) (here \( \mathbb{N} \) denotes the set of nonnegative integers) that is closed under addition, contains the zero element, and its complement in \( \mathbb{N} \) is finite.

If \( A \) is a nonempty subset of \( \mathbb{N} \), we denote by \( (A) \) the submonoid of \( \mathbb{N} \) generated by \( A \), that is,

\[
(A) = \{\lambda_1n_1 + \cdots + \lambda_en_e \mid n_1, \ldots, n_e \in \mathbb{N}, \lambda_1, \ldots, \lambda_e \in A\}.
\]

It is well known (see for instance [41, 45]) that \( (A) \) is a numerical semigroup if and only if \( \gcd(A) = 1 \).

If \( S \) is a numerical semigroup and \( S = (A) \) for some \( A \subseteq S \), then we say that \( A \) is a system of generators of \( S \) or that \( A \) generates \( S \). Moreover, \( A \) is a minimal system of generators of \( S \) if no proper subset of \( A \) generates \( S \). In [41] it is shown that every numerical semigroup admits a unique minimal system of generators, and it has finitely many elements.

Let \( S \) be a numerical semigroup and let \( \{n_1 < n_2 < \cdots < n_k\} \) be its minimal system of generators. The integers \( n_k \) and \( e \) are known as the multiplicity and embedding dimension of \( S \), and we will refer to them by using \( m(S) \) and \( e(S) \), respectively. This notation might seem amazing, but it is not so if one takes into account that there exists a large list of manuscripts devoted to the study of analytically irreducible one-dimensional local domains via their value semigroups, which are numerical semigroups. The invariants we just introduced, together with others that will show up later in this work, have an interpretation in that context, and this is why they have been named in this way. Along this line, [31] is a good reference for the translation for the terminology used in the Theory of Numerical Semigroups and Algebraic Geometry.

Frobenius (1849–1917) during his lectures proposed the problem of giving a formula for the greatest integer that is not representable as a linear combination, with nonnegative integer coefficients, of a fixed set of integers with greatest common divisor equal to 1. He also raised the question of determining how many positive integers do not admit such a representation. With our terminology, the first problem is equivalent to that of finding a formula in terms of the generators of a numerical semigroup \( S \) of the greatest integer not belonging to \( S \) (recall that its complement in \( \mathbb{N} \) is finite). This number is thus known in the literature as the Frobenius number of \( S \), and we will denote it by \( F(S) \). The elements of \( H(S) = \mathbb{N} \setminus S \) are called gaps of \( S \). Therefore the second problem consists in determining the cardinality of \( H(S) \), sometimes known as genus of \( S \) ([25]) or degree of singularity of \( S \) ([13]).

In [62] Sylvester solves the just quoted problems of Frobenius for embedding dimension two. For semigroups with embedding dimension greater than or equal to three these problems remain open. The current state of the problem is quite well collected in [36].

Let \( S \) be a numerical semigroup. Following the terminology introduced in [36] an integer \( x \) is said to be a pseudo-Frobenius number of \( S \) if \( x \not\in S \) and \( x + S \cap \{a\} \subseteq S \). We will denote by \( PF(S) \) the set of pseudo-Frobenius numbers of \( S \). The cardinality of \( PF(S) \) is called the type of \( S \) (see [31]) and we will denote it by \( t(S) \). It is proved in [18] that if \( e(S) = 2 \), then \( t(S) = 1 \), and if \( e(S) = 3 \), then \( t(S) \leq 1 \). It is also shown that if \( e(S) \geq 4 \), then \( t(S) \) can be arbitrarily large, \( t(S) \leq m(S) - 1 \) and that \( (t(S) - 1)g(S) \leq t(0)(F(S) + 1) \). This is the starting point of a new line of research that consists in trying to determine the type of a numerical semigroup, once other invariants like multiplicity, embedding dimension, genus or Frobenius number are fixed.

Wilf in [66] conjectures that if \( S \) is a numerical semigroup, then \( e(S)g(S) \leq (e(S) - 1)(F(S) + 1) \). Some fami-
lies of numerical semigroups for which it is known that the conjecture is true are collected in [16]. Other such families can be seen in [33,59]. The general case remains open.

Bras-Amorós computes in [5] the number of numerical semigroups with genus \( g \in \{0,\ldots,9\} \), and conjectures that the growth is similar to that of Fibonacci’s sequence. However it has not been proved yet that there are more semigroups of genus \( g \) than of genus \( g+1 \). Several attempts already appear in the literature. Kaplan [23] uses an approach that involves counting the semigroups by genus and multiplicity. He poses many related conjectures which could be taken literally and be proposed here as problems. We suggest them to the reader. A different approach, dealing with the asymptotical behavior of the sequence of the number of numerical semigroups by genus, has been followed by Zhao [69]. Some progress has been achieved by Zhai [68], but many questions remain open.

2. Proportionally Modular Semigroups

Following the terminology introduced in [32], a proportionally modular Diophantine inequality is an expression of the form \( ax + by \leq cx + dy \), with \( a, b \) and \( c \) positive integers.

These semigroups have been wide-

ly studied. Their main motivation comes from a work by Kunz [26] from which it can be deduced that a numerical semigroup is symmetric if and only if it is the quotient of some proportionally modular Diophantine equation. It is proved in [32] by proving that a numerical semigroup is proportionally modular if and only if it is of the form \( S = (a + \alpha + \beta I)/d \) with \( a, d \) positive integers. We still do not have formulas for \( F(a,a+\alpha+\beta I)/d \), \( g(a,a+\alpha+\beta I)/d \), \( m(a,a+\alpha+\beta I)/d \), \( t((a,a+\alpha+\beta I)/d \) and \( e(a,a+\alpha+\beta I)/d \). The next step in this line of research would be studying those numerical semigroups that are the quotient of a numerical semigroup with embedding dimension three by a positive integer. Unfortunately we do not have a procedure that allows us to distinguish such a semigroup from the rest. Moreover, we still do not know of any example of semigroups that are not of this form.

A numerical semigroup \( S \) is symmetric if \( S \in \mathbb{Z} \), \( S \) implies \( F(S) = x \in S \). These semigroups have been wide-

ly studied. Their main motivation comes from a work by Kunz [26] from which it can be deduced that a numerical semigroup is symmetric if and only if it is the quotient of some proportionally modular Diophantine equation. It is proved in [32] by proving that a numerical semigroup is proportionally modular if and only if it is of the form \( S = (a + \alpha + \beta I)/d \) with \( a, d \) positive integers. We still do not have formulas for \( F(a,a+\alpha+\beta I)/d \), \( g(a,a+\alpha+\beta I)/d \), \( m(a,a+\alpha+\beta I)/d \), \( t((a,a+\alpha+\beta I)/d \) and \( e(a,a+\alpha+\beta I)/d \). The next step in this line of research would be studying those numerical semigroups that are the quotient of a numerical semigroup with embedding dimension three by a positive integer. Unfortunately we do not have a procedure that allows us to distinguish such a semigroup from the rest. Moreover, we still do not know of any example of semigroups that are not of this form.

A numerical semigroup \( S \) is symmetric if \( S \in \mathbb{Z} \), \( S \) implies \( F(S) = x \in S \). These semigroups have been wide-

ly studied. Their main motivation comes from a work by Kunz [26] from which it can be deduced that a numerical semigroup is symmetric if and only if it is the quotient of some proportionally modular Diophantine equation. It is proved in [32] by proving that a numerical semigroup is proportionally modular if and only if it is of the form \( S = (a + \alpha + \beta I)/d \) with \( a, d \) positive integers. We still do not have formulas for \( F(a,a+\alpha+\beta I)/d \), \( g(a,a+\alpha+\beta I)/d \), \( m(a,a+\alpha+\beta I)/d \), \( t((a,a+\alpha+\beta I)/d \) and \( e(a,a+\alpha+\beta I)/d \). The next step in this line of research would be studying those numerical semigroups that are the quotient of a numerical semigroup with embedding dimension three by a positive integer. Unfortunately we do not have a procedure that allows us to distinguish such a semigroup from the rest. Moreover, we still do not know of any example of semigroups that are not of this form.

A numerical semigroup \( S \) is symmetric if \( S \in \mathbb{Z} \), \( S \) implies \( F(S) = x \in S \). These semigroups have been wide-

ly studied. Their main motivation comes from a work by Kunz [26] from which it can be deduced that a numerical semigroup is symmetric if and only if it is the quotient of some proportionally modular Diophantine equation. It is proved in [32] by proving that a numerical semigroup is proportionally modular if and only if it is of the form \( S = (a + \alpha + \beta I)/d \) with \( a, d \) positive integers. We still do not have formulas for \( F(a,a+\alpha+\beta I)/d \), \( g(a,a+\alpha+\beta I)/d \), \( m(a,a+\alpha+\beta I)/d \), \( t((a,a+\alpha+\beta I)/d \) and \( e(a,a+\alpha+\beta I)/d \). The next step in this line of research would be studying those numerical semigroups that are the quotient of a numerical semigroup with embedding dimension three by a positive integer. Unfortunately we do not have a procedure that allows us to distinguish such a semigroup from the rest. Moreover, we still do not know of any example of semigroups that are not of this form.

A numerical semigroup \( S \) is symmetric if \( S \in \mathbb{Z} \), \( S \) implies \( F(S) = x \in S \). These semigroups have been wide-

ly studied. Their main motivation comes from a work by Kunz [26] from which it can be deduced that a numerical semigroup is symmetric if and only if it is the quotient of some proportionally modular Diophantine equation. It is proved in [32] by proving that a numerical semigroup is proportionally modular if and only if it is of the form \( S = (a + \alpha + \beta I)/d \) with \( a, d \) positive integers. We still do not have formulas for \( F(a,a+\alpha+\beta I)/d \), \( g(a,a+\alpha+\beta I)/d \), \( m(a,a+\alpha+\beta I)/d \), \( t((a,a+\alpha+\beta I)/d \) and \( e(a,a+\alpha+\beta I)/d \). The next step in this line of research would be studying those numerical semigroups that are the quotient of a numerical semigroup with embedding dimension three by a positive integer. Unfortunately we do not have a procedure that allows us to distinguish such a semigroup from the rest. Moreover, we still do not know of any example of semigroups that are not of this form.

A numerical semigroup \( S \) is symmetric if \( S \in \mathbb{Z} \), \( S \) implies \( F(S) = x \in S \). These semigroups have been wide-

ly studied. Their main motivation comes from a work by Kunz [26] from which it can be deduced that a numerical semigroup is symmetric if and only if it is the quotient of some proportionally modular Diophantine equation. It is proved in [32] by proving that a numerical semigroup is proportionally modular if and only if it is of the form \( S = (a + \alpha + \beta I)/d \) with \( a, d \) positive integers. We still do not have formulas for \( F(a,a+\alpha+\beta I)/d \), \( g(a,a+\alpha+\beta I)/d \), \( m(a,a+\alpha+\beta I)/d \), \( t((a,a+\alpha+\beta I)/d \) and \( e(a,a+\alpha+\beta I)/d \). The next step in this line of research would be studying those numerical semigroups that are the quotient of a numerical semigroup with embedding dimension three by a positive integer. Unfortunately we do not have a procedure that allows us to distinguish such a semigroup from the rest. Moreover, we still do not know of any example of semigroups that are not of this form.

A numerical semigroup \( S \) is symmetric if \( S \in \mathbb{Z} \), \( S \) implies \( F(S) = x \in S \). These semigroups have been wide-

ly studied. Their main motivation comes from a work by Kunz [26] from which it can be deduced that a numerical semigroup is symmetric if and only if it is the quotient of some proportionally modular Diophantine equation. It is proved in [32] by proving that a numerical semigroup is proportionally modular if and only if it is of the form \( S = (a + \alpha + \beta I)/d \) with \( a, d \) positive integers. We still do not have formulas for \( F(a,a+\alpha+\beta I)/d \), \( g(a,a+\alpha+\beta I)/d \), \( m(a,a+\alpha+\beta I)/d \), \( t((a,a+\alpha+\beta I)/d \) and \( e(a,a+\alpha+\beta I)/d \). The next step in this line of research would be studying those numerical semigroups that are the quotient of a numerical semigroup with embedding dimension three by a positive integer. Unfortunately we do not have a procedure that allows us to distinguish such a semigroup from the rest. Moreover, we still do not know of any example of semigroups that are not of this form.
ry numerical semigroup is one half of an infinite number of symmetric numerical semigroups. The apparent parallelism between symmetric and pseudo-symmetric numerical semigroups fails as we can see in [17], where it is proved that a numerical semigroup is irreducible if and only if it is one half of a pseudo-symmetric numerical semigroup. As a consequence we have that every numerical semigroup is a quarter of infinitely many pseudo-symmetric numerical semigroups. In [61], it is also shown that for every positive integer $i$ and every numerical semigroup $S$, there exist infinitely many symmetric numerical semigroups $T$ such that $S = T / i$, and if $i \geq 1$, then there exist infinitely many pseudo-symmetric numerical semigroups $T$ with $S = T / i$.

From the definition, we deduce that a numerical semigroup $S$ is symmetric if and only if there exist positive integers $p$, $q$, $m$, and $n$, and only if it $g(S) = (p \mathbb{Z} + q \mathbb{Z}) / m \mathbb{Z}$.

The cardinality of the set of fundamental gaps of a numerical semigroup is an invariant of the semigroup. We can therefore open a new line of research by studying numerical semigroups attending to their number of fundamental gaps. It would be also interesting to find simple sufficient conditions that allow us to decide when a subset $X$ of $\mathbb{N}$ is the set of fundamental gaps of some numerical semigroup.

Let $S$ be a numerical semigroup. In [13] the set $S$ of all numerical semigroups such that $S = T / 2$ is studied, the semigroup of the “doubles” of $S$. In the just quoted work we raise the question of finding a formula that depends on $S$ and allows us to compute the minimum of the Frobenius numbers of the doubles of $S$.

Following this line we can ask ourselves about the set of all “triples” (or multiples in general) of a numerical semigroup.

Finally, it would be interesting to characterize the families of numerical semigroups satisfying that any of its elements can be realized as a quotient of some element of the family by a fixed positive integer.

4. Frobenius Varieties

A directed graph $G$ is a pair $(V, E)$, where $V$ is a nonempty set whose elements are called vertices, and $E$ is a subset of $(\{v, w\} \in V \times V \mid v \neq w \}$. The elements of $E$ are called edges of the graph. A path connecting two vertices $v$ and $w$ is a sequence of distinct vertices of the form $(v_0, v_1, v_2, \ldots, v_n, w)$ with $v_0 = v$ and $v_n = w$. A graph $G$ is a tree if there exists a vertex $r$ (called the root of $G$) such that for any other vertex $v$ of $G$, there exists an unique path connecting $v$ and $r$. If $(x, y)$ is an edge of the tree, then $x$ is a son of $y$. A vertex of a tree is a leaf if it has no sons.

Let $\mathcal{G}$ be the set of all numerical semigroups. We define the graph associated to $\mathcal{G}$, $(\mathcal{G}, \mathcal{E})$, to be the graph whose vertices are all the elements of $\mathcal{G}$ and $(T, S) \in \mathcal{E}$ if and only if $T$ is an intersection of irreducible modular semigroups such that $T \subseteq S$.

These results allow us to construct recursively the set of numerical semigroups starting with $N$.

\[
N \rightarrow (1) \rightarrow (1, 4) \rightarrow (1, 4, 6) \rightarrow (1, 4, 6, 7) \rightarrow \ldots
\]

The level of a vertex in a directed graph is the length of the path connecting this vertex with the root. Note that in $(\mathcal{G}, \mathcal{E})$ the level of a vertex coincides with its genus as a numerical semigroup. Therefore, the Bra-Namor’s conjecture is true in the end of the first section can be reformulated by saying that in $(\mathcal{G}, \mathcal{E})$ there are more vertices in the $(n + 1)$-th level than in the $n$-th one.

A Frobenius variety is a nonempty family $\mathcal{P}$ of numerical semigroups such that

1) If $T \subseteq \mathcal{P}$, then $S \cap T \subseteq \mathcal{P}$.

2) If $S \subseteq \mathcal{P}$, $S \neq N$, then $S \cap (\mathbb{N} / i) \subseteq \mathcal{P}$.

The concept of Frobenius variety was introduced in [38] with the aim of generalizing most of the results in [6, 14, 48, 49]. In particular, the semigroups that belong to a Frobenius variety can be arranged as a directed tree with similar properties to those of $(\mathcal{G}, \mathcal{E})$.

Clearly, $\mathcal{P}$ is a Frobenius variety. If $A \subseteq \mathcal{P}$, then $A \subseteq \mathcal{P}$ is also a Frobenius variety. In particular, $(\mathcal{G}, \mathcal{P})$, the set of all numerical semigroups that contain $S$, is a Frobenius variety. We next give some interesting examples of Frobenius varieties.

Inspired by [1], Lipman introduces and motivates in [17] the study of Arf rings. The characterization of them via their numerical semigroup of values, brings us to the following concept: a numerical semigroup $S$ is said to be Arf if for every $x, y \in S$, with $x, y > 2$, we have that $x + y - z \in S$. It is proved in [48] that the set of all Arf numerical semigroups is a Frobenius variety.

Saturated rings were introduced independently in three distinct ways by Zariski ([67]), Pham-Teissier ([19]) and Campillo ([25]), although the definitions given in these works are equivalent on algebraically closed fields of characteristic zero. LIKE the case of numerical semigroups with the Arf property, saturated numerical semigroups appear when characterizing these rings in terms of their numerical semigroups of values. A numerical semigroup $S$ is saturated if for every $s_1, s_2, \ldots, s_n \in S$, and for all $i \in \{1, \ldots, n\}$, $s_i, s_{i+1}, \ldots, s_n \in \mathbb{Z}$ being integers such that $s_i + s_{i+1} + \cdots + s_n \geq 0$, then we have $s_i + s_{i+1} + \cdots + s_n \in S$. It is proved in [48] that the set of saturated numerical semigroups is a Frobenius variety.
$a_1 s_1 + \cdots + a_n s_n \in S$. We denote by $\mathcal{F}$ the set of all numerical semigroups that admit a pattern $P$. Then the set of numerical semigroups with the Arf property is $\mathcal{F}_{\text{Arf}}$. It is proved in [6] that for every pattern $P$ of a special type (strongly admissible), $\mathcal{F}$ is a Frobenius variety. What varieties arise in this way? It would be interesting to give a weaker definition of pattern such that every variety becomes the variety associated to a pattern.

The intersection of Frobenius varieties is again a Frobenius variety. This fact allows us to construct new Frobenius varieties from known Frobenius varieties and more generally, it allows us to talk of the Frobenius variety generated by a family $X$ of numerical semigroups. This variety will be denoted by $\mathcal{F}(X)$, and it is defined to be the intersection of all Frobenius varieties containing $X$. If $X$ is finite, then $\mathcal{F}(X)$ is finite and it is shown in [38] how to compute all the elements of $\mathcal{F}(X)$.

Let $\mathcal{F}$ be a Frobenius variety. A submonoid $M$ of $\mathcal{N}$ is a $\mathcal{F}$-monoid if it can be expressed as an intersection of elements of $\mathcal{F}$. It is clear that the intersection of $\mathcal{F}$-monoids is again a $\mathcal{F}$-monoid. Thus given $A \subseteq \mathcal{N}$ we can define the $\mathcal{F}$-monoid generated by $A$ as the intersection of all $\mathcal{F}$-monoids containing $A$. We will denote by $\mathcal{F}(A)$ this $\mathcal{F}$-monoid and we will say that $A$ is a $\mathcal{F}$-system of generators of $\mathcal{F}$. If there is no proper subset of $A$ being a $\mathcal{F}$-system of generators of $\mathcal{F}(A)$, then $A$ is a minimal $\mathcal{F}$-system of generators of $\mathcal{F}(A)$. It is proved in [38] that every $\mathcal{F}$-monoid admits an unique minimal $\mathcal{F}$-system of generators, and that moreover this system is finite.

We define the directed graph $\mathcal{G}(\mathcal{F})$, in the same way we defined $\mathcal{G}(\mathcal{S})$, that is, as the graph whose vertices are the elements of $\mathcal{F}$, and $(T,S) \in \mathcal{G}(\mathcal{F})$ is an edge of the above graph if $S = T \cup (T,F)$. This graph is a tree with root $\mathcal{N}$ ([38]). Moreover, the sons of a semigroup $S$ in $\mathcal{F}$ are $S \setminus \{s_1, \ldots, s_k\}$, where $s_1, \ldots, s_k$ are the minimal $\mathcal{F}$-generators of $S$ greater than $F(S)$. This fact allows us to find all the elements of the variety $\mathcal{F}$ from $\mathcal{N}$.

Figure 1 represents part of the tree associated to the variety of numerical semigroups with the Arf property. Figure 2 represents part of the tree corresponding to saturated numerical semigroups.

As a generalization of Bras-Amorós’ conjecture, we can raise the following question. If $\mathcal{F}$ is a Frobenius variety, does there exist on $\mathcal{G}(\mathcal{F})$ more vertices in the $(n + 1)$th level than in the $n$th one? The answer to this question is no, as it is proved in [38, Example 26]. However, the same question in the case of $\mathcal{F}$ being infinite remains open. Another interesting question would be characterizing those Frobenius varieties that verify the Bras-Amorós’ conjecture.

If $\mathcal{F}$ is a Frobenius variety and $S \in \mathcal{F}$, then it is known that $S$ admits an unique minimal $\mathcal{F}$-system of generators, and moreover it is finite. The cardinality of the set above is an invariant of $S$ that will be called the embedding $\mathcal{F}$-dimension of $S$, and it will be denoted by $\mathcal{e}_\mathcal{F}(S)$. As a generalization of Wilf’s conjecture, we would like to characterize those Frobenius varieties $\mathcal{F}$ such that for every $S \in \mathcal{F}$, then $\mathcal{e}_\mathcal{F}(S) = (\mathcal{e}_\mathcal{F}(S) - 1)((\mathcal{F}(S) + 1)$.

Clearly, the Frobenius variety generated by irreducible numerical semigroups is $\mathcal{F}_0$, the set of all numerical semigroups. What is the Frobenius variety generated only by the symmetric ones? and by the pseudo-symmetric ones?

5. PRESENTATIONS OF A NUMERICAL SEMIGROUP

Let $(S, +)$ be a commutative monoid. A congruence $\sigma$ over $S$ is an equivalence relation that is compatible with addition, that is, if $a \equiv b \pmod{\sigma}$ and $c \equiv d \pmod{\sigma}$ then $a + c \equiv b + d \pmod{\sigma}$ for all $a, b, c, d \in S$. The set $S^{\sigma}$ endowed with the operation $[a] + [b] = [a + b]$ is a monoid. We will call it the quotient monoid of $S$ by $\sigma$.

If $S$ is generated by $\{s_1, \ldots, s_n\}$, then the map $\varphi: \mathbb{N}^n \rightarrow S$, $(a_1, \ldots, a_n) \mapsto a_1 s_1 + \cdots + a_n s_n$ is a monoid epimorphism. Therefore $S$ is isomorphic to $\mathbb{N}_0^n$, where $a \equiv b \pmod{\sigma}$ if and only if $\varphi(a) = \varphi(b)$. The intersection of congruences over a monoid $S$ is again a congruence over $S$. This fact allows us, given $\sigma \subseteq S \times S$, to define the concept of congruence generated by $\sigma$ as the intersection of all congruences over $S$ containing $\sigma$, and it will be denoted by $[\sigma]$.

Rédéi proves in [31] that every congruence over $\mathbb{N}$ is finitely generated, that is, there exists a subset of $\mathbb{N} \times \mathbb{N}$ with finitely many elements generating it. As a consequence we have that giving a finitely generated monoid is, up to isomorphism, equivalent to giving a finite subset of $\mathbb{N} \times \mathbb{N}$.

If $S$ is a numerical semigroup with minimal generators system $\{n_1, \ldots, n_k\}$, then there exists a finite subset $\sigma$ of $\mathbb{N}_0 \times \mathbb{N}_0$ such that $S$ is isomorphic to $\mathbb{N}_0^{\sigma}(\sigma)$. We say that $\sigma$ is a presentation of $S$. If moreover $\sigma$ has the least possible cardinality, then $\sigma$ is a minimal presentation of $S$.

A (non directed) graph $G$ is a pair $(V, E)$, where $V$ is a nonempty set of elements called vertices, and $E$ is a subset of $\{[u, v] \mid u, v \in V, u \neq v\}$. The non ordered pair $[u, v]$ will be denoted by $uv$, and if it belongs to $E$, then we say that $uv$ is an edge of $G$. A sequence of the form $v_0 v_1 v_2 \cdots v_m$ is a path of length $m$ connecting the vertices $v_0$ and $v_m$. A graph is connected if any two distinct vertices are connected by a path. A connected graph $G' = (V', E')$ is said to be a subgraph of $G$ if $V' \subseteq V$ and $E' \subseteq E$. A connected component of $G$ is a maximal connected subgraph of $G$. It is well known (see for instance [28]) that a connected graph with $n$ vertices has at least $n - 1$ edges. A (finite) tree with $n$ vertices is a connected graph with $n - 1$ edges.
Let us remind now the method described in [33] for computing the minimal presentation of a numerical semifield. Let \( S \) be a numerical semifield with minimal generating system \((n_1, \ldots, n_r)\). For each \( n \in S \), let us define \( G_n = (V_n, E_n) \), where \( V_n = \{ \{ n \}, \{ n - n, \} \} \) and \( E_n = \{ \{ n \}(n, n - n) \in S \} \). If \( G_n \) is connected, let \( v_n = 0 \). If \( G_n \) is not connected and \( V_n \rightarrow V \), are the sets of vertices corresponding to the connected components in \( G_n \), then \( V = \{ v_n \mid n \in S \} \). The minimal system of generators of \((n_1, \ldots, n_r, v_n)\), where \( n \in G_n \) and \( v \) is the root of zero when \( v_n \in V \). It is proved in [35] that all minimal presentations are isomorphic to each other.

It is proved in [45] how given minimal presentations of \( S \) can be expressed as the quotient of a complete intersection by a positive integer? What is the least Frobenius number of a numerical semigroup?

We note now the method described in [35] for computing the minimal presentation of a numerical semifield. Let us observe that for any integer \( n \geq 1 \), one easily gets a minimal presentation of \( \mathbb{Z}_n \), with \( \mathbb{Z}_n \) being a symmetric numerical semigroup. More specifically, we suggest the following problem: given \( \sigma = ((c_0, 0, \ldots, 0), \ldots, (c_{n-1}, 0, \ldots, 0)) \), which conditions the integers \( c_0, \ldots, c_{n-1} \) have to verify so that \( \mathbb{Z}_{\langle \sigma \rangle} \) is isomorphic to a numerical semifield?

Hertog proved in [21] that embedding dimension three numerical semifields always have a minimal presentation of this form. Next numerical semifields introduced by Komeda in [24] are also of this form.

### 6. Numerical Semifields with Embedding Dimension Three

Hertog proves in [21] that a numerical semifield with embedding dimension three is symmetric if and only if it is a complete intersection. This fact allows us to characterize symmetric numerical semifields with embedding dimension three in the following way (see [41]).

A numerical semifield \( S \) with \( e(S) = 1 \) is symmetric if and only if it is a complete intersection. This conclusion is true if and only if \( S = (am, bm, cm) \), with \( a, b, c, m \) nonnegative integers, such that \( m \equiv a + b + c \equiv 0 \) and \( gcd(m, m_1, m_2) = gcd(a, b, m) \).

It is proved in [18] that \( \mathbb{Z}_n \) is a symmetric numerical semifield with \( e(S) = 1 \). Moreover, this is the case if \( n = \frac{a(n_1) - 1}{\langle \sigma \rangle} \in \mathbb{Z}_n \) and \( n \leq 1 \), which is a result proved in [42].

The set of lengths of \( S \) is minimally generated by \( \{ n_1, n_2, \ldots, n_r \} \) and \( n \) being pairwise relatively prime. For each \( i \in \{ 1, 2, \ldots, r \} \), let \( c_i = \min \{ n \in \mathbb{N} \mid \langle \sigma \rangle \in \mathbb{Z}_n \} \). Moreover, in this case, \( S = \{ \langle \sigma \rangle \} \).

In [42] the authors give formulas for the Frobenius number of \( S \) and the genus of the numerical semifield \( \{ n_1, n_2, \ldots, n_r \} \) from the solutions of the above system. Thus it seems natural to ask, given positive integers \( r, t \in \{ 1, 2, \ldots, r \} \), when \( r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 + r_9 + r_{10} \) are pairwise relatively prime?

Let \( n \) be a numerical semifield minimally generated by three positive integers \( n_1, n_2, \) and \( n_3 \) being pairwise relatively prime. For each \( i \in \{ 1, 2, 3 \} \), let \( c_i = \min \{ n \in \mathbb{N} \mid \langle \sigma \rangle \in \mathbb{Z}_n \} \). Moreover, in this case, \( S = \{ \langle \sigma \rangle \} \).

Let us observe that for any integer \( n \geq 1 \), one easily gets a minimal presentation of \( \mathbb{Z}_n \), with \( \mathbb{Z}_n \) being a symmetric numerical semigroup. More specifically, we suggest the following problem: given \( \sigma = ((c_0, 0, \ldots, 0), \ldots, (c_{n-1}, 0, \ldots, 0)) \), which conditions the integers \( c_0, \ldots, c_{n-1} \) have to verify so that \( \mathbb{Z}_{\langle \sigma \rangle} \) is isomorphic to a numerical semifield?

Hertog proved in [21] that embedding dimension three numerical semifields always have a minimal presentation of this form. Next numerical semifields introduced by Komeda in [24] are also of this form.

### 7. Non-uniform Factorization Invariants

Let \( S \) be a numerical semifield minimally generated by \( \{ n_1, \ldots, n_r \} \). Then we already know that \( S \) is isomorphic to \( \mathbb{Z}^{n_r} \), where \( \mathbb{Z}^{n_r} \) is the kernel congruence of the epimorphism \( \varphi : \mathbb{Z}^{n_r} \rightarrow S \). The set of \( n \) is minimally generated by \( \{ n_1, \ldots, n_r \} \) from the solutions of the above system. Thus it seems natural to ask, given positive integers \( r, t \in \{ 1, 2, \ldots, r \} \), when \( r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 + r_9 + r_{10} \) are pairwise relatively prime?

Let \( S \) be a numerical semifield minimally generated by three positive integers \( n_1, n_2, \) and \( n_3 \) being pairwise relatively prime. For each \( i \in \{ 1, 2, 3 \} \), let \( c_i = \min \{ n \in \mathbb{N} \mid \langle \sigma \rangle \in \mathbb{Z}_n \} \). Moreover, in this case, \( S = \{ \langle \sigma \rangle \} \).

For each \( i \in \{ 1, 2, 3 \} \), let \( c_i = \min \{ n \in \mathbb{N} \mid \langle \sigma \rangle \in \mathbb{Z}_n \} \). Moreover, in this case, \( S = \{ \langle \sigma \rangle \} \).

Let us observe that for any integer \( n \geq 1 \), one easily gets a minimal presentation of \( \mathbb{Z}_n \), with \( \mathbb{Z}_n \) being a symmetric numerical semigroup. More specifically, we suggest the following problem: given \( \sigma = ((c_0, 0, \ldots, 0), \ldots, (c_{n-1}, 0, \ldots, 0)) \), which conditions the integers \( c_0, \ldots, c_{n-1} \) have to verify so that \( \mathbb{Z}_{\langle \sigma \rangle} \) is isomorphic to a numerical semifield?

Hertog proved in [21] that embedding dimension three numerical semifields always have a minimal presentation of this form. Next numerical semifields introduced by Komeda in [24] are also of this form.

### 6. Numerical Semifields with Embedding Dimension Three

Hertog proves in [21] that a numerical semifield with embedding dimension three is symmetric if and only if it is a complete intersection. This fact allows us to characterize symmetric numerical semifields with embedding dimension three in the following way (see [41]).
the set of differences of lengths of factorizations of \( \mathcal{S} \) is 
\[ \Delta(\mathcal{S}) = \{ n \in \mathbb{N} : \exists x, y, z, \ldots \in \mathcal{S} \text{ such that } n = d(x, y) + d(y, z) + \cdots \}. \]
Moreover \( \Delta(\mathcal{S}) = \bigcup_{n \in \mathbb{N}} \Delta(n) \).

These sets are known to be eventually periodic \([12]\).

The invariant \( \Delta(\mathcal{S}) \) is the least positive integer \( n \) such that \( d(x, y) \equiv d(y, z) \equiv \cdots \equiv d(z_{n-1}, z_n) \equiv 0 \pmod{n} \).

The tame degree of \( \mathcal{S} \) is 
\[ \deg \mathcal{S} = \min \{ n \in \mathbb{N} : \Delta(n) = \emptyset \}. \]

The catenary degree of \( \mathcal{S} \) is 
\[ \gamma \mathcal{S} = \min \{ n \in \mathbb{N} : \Delta(n) \neq \emptyset \}. \]

Another problem proposed by A. Geroldinger is to find an element in it with a given set of lengths.

References


1. Introduction

“There is strong shadow where there is much light”
Goethe in Götz von Berlichingen

1.1 The basic framework

In order to start playing with dynamical systems we need a place to play and a given rule acting on it. Once we establish that, we wonder what happens when we repeat the rule ad infinitum. We are mainly interested in two types of playgrounds: volume manifolds and symplectic manifolds. On volume-manifolds the rule is the action of a volume-preserving diffeomorphism, and on symplectic manifolds the rule is the action of a symplectomorphism. Let us now formalize these concepts.

Let $M$ stand for a closed, connected and $C^\omega$ Riemannian manifold of dimension $d \geq 1$ and let $\nu$ be a volume-form on $M$. Once we equip $M$ with $\nu$ we denominate it by a volume-manifold. By a classic result by Moser (see [20]) we know that, in brief terms, there is only one volume-form on $M$.

Denote by $M$ a $d$-dimensional $(d \geq 1)$ manifold with a Riemannian structure and endowed with a closed and nondegenerate $\omega$-form $\omega$ called symplectic form. Let $\nu$ stand for the volume measure associated to the volume form $\omega$.

Let $\nu = \omega^d = \omega \wedge \ldots \wedge \omega$. By the

Abstract.—We explore uniform hyperbolicity and its relation with the pseudo orbit tracing property. This property indicates that a sequence of points which is nearly an orbit (affected with a certain error) may be shadowed by a true orbit of the system. We obtain that, when a conservative map has the shadowing property and, moreover, all the conservative maps in a $C^\omega$-small neighborhood display the same property, then the map is globally hyperbolic.

MSC 2000: primary 37D20, 37C50; secondary 37C05, 37J10.

Keywords.—Volume-preserving maps; pseudo-orbits; shadowing; hyperbolicity.

Tracing orbits on conservative maps

by Mário Bessa*

* Universidade da Beira Interior, Covilhã, Portugal
bessa@ubi.pt